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THE POYNTING-SWIFT EFFECT IN RELATION TO INITIAL AND POST-YIELD DEFORMATION

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Abstract—The material behaviour predicted by the constitutive equation of a general elastic material is considered in relation to the combined extension and torsion of a thin-walled tube and a solid rod. This state of combined stressing is considered in relation to material behaviour at small strains, and at finite strains in the context of initial yield and the Poynting-Swift effect. A composite yield condition has been postulated on general grounds which, over a given range of application, is identical to the von Mises yield criterion, or some modified form of it, and over the remaining range of application takes the form of a 12-sided, linear, piece-wise continuous yield condition. The theory is applied to the classical combined-stress experiments of Taylor and Quinney[3], using a total deformation-type constitutive equation. It is shown that the proposed yield condition when combined with the new constitutive equation, is in full agreement with the useable experimental data obtainable from these particular combined–stress measurements. The available studies of the Poynting-Swift effect are shown to be in general accord with the proposed yield function and total deformation type constitutive equation. In particular, for sufficiently small shear strains it has been shown that simple torsion of both a thin-walled tube and a solid rod is characterised by an axial elongation which is proportional to the square of the twist. It is also shown that the available experimental results are in quantitative agreement with the predictions of the proposed constitutive equation.

1. INTRODUCTION

Implicit in the formulation of classical plasticity theory is the assumption that the idealised, perfectly plastic material satisfies the von Mises yield criterion[1]. However, there is a growing body of experimental evidence for the existence of materials which do not satisfy the von Mises yield criterion. Most of this evidence has been obtained by way of experiments concerned with the plastic response of thin-walled tubes, as for example those of Lode[2] and Taylor and Quinney[3]. The tubes are subjected to various combinations of axial tension or compression, torsion and internal or external pressure. It is the combined stressing of thin-walled tubes in axial tension and simple torsion in the way used by Taylor and Quinney[3] which is of present interest. There are two possible approaches to this type of experiment. They can be used to evaluate the form of the yield function without having recourse to any particular constitutive equation. This approach has two disadvantages. It requires direct measurement of changes in volume of the bore of the tubes—a measurement which remains the most difficult to make within an acceptable degree of accuracy. The second limitation is that it requires direct observation of the yield stress in simple tension and in simple torsion. There is no method by which both yield values can be independently measured to an acceptable degree of accuracy. In the measurements of Taylor and Quinney[3], the onset of plastic distortion on reloading in torsion can only be evaluated by empirical methods. The present article is concerned, in part, with an alternative approach. Assuming that the form of the constitutive equation for the purely elastic behaviour is known, a method has been developed for determining the initial yield function from combined stress measurements of the type described by Taylor and Quinney[3]. This approach has the advantage that it does not require an independent measurement of the yield stress in simple torsion.

For solids the mechanical response of a rod or tube twisted in simple torsion is characterised by certain nonlinear normal stress effects. These are the Poynting[4, 5] effect in the elastic range of deformation and the similar Swift[6] effect in the plastic range of deformation. The Poynting effect relates to the observation by Poynting that the length of various steel, copper and brass wires increased when twisted in the elastic range, and that the elongation was proportional to the square of the twist. Swift[6] observed that a permanently twisted rod elongates in the direction of the principal axis about which the rod is being twisted, and this permanent elongation increases with increasing shear strain. Although there is general agreement between the results of the experimental studies of the Poynting-Swift effect and the behaviour predicted by a proposed (Billington[7]) constitutive equation of a general elastic material, there remains the question of the extent to which the experimental work is in quantitative agreement with these predictions. It is this aspect of the mechanical response of an incompressible solid with which the present study is, in part, concerned.

2. EXTENSION AND TORSION OF A THIN-WALLED TUBE AND A SOLID ROD

2.1 *The deformation and stress ratios*

Using standard notation and conventions, the constitutive equation for a particular class of incompressible, elastic materials can be expressed in the form[7]

$$\mathbf{G} = \mathbf{T} + b\mathbf{T}^2 = -P\mathbf{I} + \phi_1\mathbf{B} + \phi_2\mathbf{B}^2,$$

which can be expressed in the form

$$\mathbf{G}'(\mathbf{T}') = \mathbf{M}'(\mathbf{B}') = \beta_1\mathbf{B}' + \beta_{-1}(\mathbf{B}^{-1})', \tag{2.1}$$

where a prime denotes a deviator.

Consider a right-circular tube or solid-rod of incompressible material. Let (R, Θ, Z) be the cylindrical referential coordinates in the initial state of a particle that is located in the deformed configuration by the cylindrical spatial coordinates (r, θ, z).

For a twisted tube with its principal axis aligned with the Z-axis, the non-vanishing physical components $T\langle ij\rangle$ of stress are:

$$[T\langle ij\rangle] = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & \tau_{\theta z} \\ 0 & \tau_{z\theta} & \sigma_{zz} \end{bmatrix}, \qquad (\tau_{\theta z} = \tau_{z\theta}). \tag{2.2}$$

The components of the loading function

$$\mathbf{G}' = \delta_0\mathbf{I} + \delta_1\mathbf{T}' + \delta_2\mathbf{T}'^2 = \frac{\partial h}{\partial \mathbf{T}'}, \tag{2.3}$$

follow from eqns (2.2) and (2.3):

$$[G'\langle ij\rangle = \Lambda\tau_{\theta z} \begin{bmatrix} \frac{\bar{\mu}}{\mu}\zeta & 0 & 0 \\ 0 & -\frac{1}{2}\left(\xi + \frac{\bar{\mu}}{\mu}\zeta\right) & 1 \\ 0 & 1 & \frac{1}{2}\left(\xi - \frac{\bar{\mu}}{\mu}\xi\right) \end{bmatrix} \tag{2.4}$$

where the stress ratios ξ and ζ are defined by the relations

$$\xi = \frac{\sigma_{zz} - \sigma_{\theta\theta}}{\tau_{\theta z}} = \frac{G'\langle zz\rangle - G'\langle\theta\theta\rangle}{G'\langle\theta z\rangle}, \tag{2.5}$$

$$\zeta = \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{zz}}{3\tau_{\theta z}} = \frac{\mu}{\bar{\mu}}\frac{G'\langle rr\rangle}{G'\langle\theta z\rangle}, \tag{2.6}$$

and where Lode's[2] stress parameter

$$\mu = \frac{3\sigma_1'}{\sigma_3' - \sigma_2'} = \frac{3\zeta}{(4 + \xi^2)^{1/2}}, \tag{2.7}$$

it being noted that

$$\bar{\mu} = \mu \left[1 - \frac{(3 + \mu^2)}{\mu^2} \left(1 - \frac{H}{\Lambda} \right) \right], \tag{2.8}$$

$$\begin{aligned} \Lambda &= H \left[1 - \frac{1}{6} (3 + \mu^2) \frac{\mu}{H} \frac{dH}{d\mu} \right] \\ &= \delta_1 - \delta_2 \sigma_1' = \delta_1 - \delta_2 \sigma_{rr}'. \end{aligned} \tag{2.9}$$

and that the σ_a' $(a = 1, 2, 3)$ are the deviators of the principal stresses, σ_a $(a = 1, 2, 3)$. In eqn (2.3), h is the stress intensity function

$$h = J_2' H(\omega) - \kappa = 0, \tag{2.10}$$

where

$$\omega = \frac{27}{4} \frac{J_3'^2}{J_2'^3} = \mu^2 \frac{(9 - \mu^2)^2}{(3 + \mu^2)^3}, \tag{2.11}$$

and where

$$J_2' = \frac{1}{2} \operatorname{tr} \mathbf{T}'^2, \qquad J_3' = \det \mathbf{T}', \tag{2.12}$$

are the second and third invariants of the deviatoric stress tensor $\mathbf{T}'$.

The simple deformations to be considered are

$$r = [(K + R^2)/F]^{1/2}, \qquad \theta = \Theta + DZ, \qquad z = FZ, \tag{2.13}$$

where D, F and K are constants subject to the condition that F and K have values such that $(K + R^2)/F > 0$ when R is in the interval $(R_1 \geq R \geq R_2)$ where R_1 and R_2 are the outer and inner radii, respectively, of an undeformed tube. Using eqn (2.13), together with the deformation gradient tensor $\mathbf{F}$, (see Section two of [7]), the non-vanishing physical components $B\langle ij\rangle$, $(B^{-1})\langle ij\rangle$ of the left Cauchy-Green deformation tensor

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \tag{2.14}$$

and its inverse $\mathbf{B}^{-1}$ can be obtained in the form:

$$[B\langle ij\rangle] = \begin{bmatrix} \frac{1}{\eta F} & 0 & 0 \\ 0 & \frac{\eta}{F}(1 + R^2D^2) & rDF \\ 0 & rDF & F^2 \end{bmatrix}, \quad [(B^{-1})\langle ij\rangle] = \begin{bmatrix} \eta F & 0 & 0 \\ 0 & \frac{F}{\eta} & -r\frac{D}{\eta} \\ 0 & -r\frac{D}{\eta} & \frac{1 + R^2D^2}{F^2} \end{bmatrix} \tag{2.15}$$

where the extension ratio

$$F = l/L, \tag{2.16}$$

is the ratio of the current length l to the undeformed length L of a tube or solid rod, and where

$$\eta = \frac{r^2}{R^2} F = 1 + \frac{K}{R^2} = \frac{v}{V}, \tag{2.17}$$

is the ratio of the current volume v to the undeformed volume V of a cylinder of radius R.

The components of the deformation function

$$\mathbf{M}' = \beta_1 \mathbf{B}' + \beta_{-1}(\mathbf{B}^{-1})', \tag{2.18}$$

follow from eqns (2.15) and (2.18):

$$[M'\langle ij\rangle] = \Phi B\langle\theta z\rangle \begin{bmatrix} \frac{\bar{\nu}}{\nu}\rho & 0 & 0 \\ 0 & -\frac{1}{2}\left(\chi + \frac{\bar{\nu}}{\nu}\rho\right) & 1 \\ 0 & 1 & \frac{1}{2}\left(\chi - \frac{\bar{\nu}}{\nu}\rho\right) \end{bmatrix}, \tag{2.19}$$

where the deformation ratios χ and ρ are defined by the relations

$$\chi = \frac{B\langle zz\rangle - B\langle\theta\theta\rangle}{B\langle\theta z\rangle} = \frac{M'\langle zz\rangle - M'\langle\theta\theta\rangle}{M'\langle\theta z\rangle} = \frac{G'\langle zz\rangle - G'\langle\theta\theta\rangle}{G'\langle\theta z\rangle} = \xi$$

$$= \frac{(F^3 - 1)}{DF^2}\frac{1}{r} - \frac{(\eta - 1)}{DF^2}\frac{1}{r} - \frac{D}{F}r, \tag{2.20}$$

$$\rho = \frac{2B\langle rr\rangle - B\langle\theta\theta\rangle - B\langle zz\rangle}{3B\langle\theta z\rangle} = \frac{\nu}{\bar{\nu}}\frac{M'\langle rr\rangle}{M'\langle\theta z\rangle} = \frac{\nu}{\bar{\nu}}\frac{G'\langle rr\rangle}{G'\langle\theta z\rangle} = \frac{\nu}{\mu}\zeta$$

$$= -\frac{1}{3}\left[\frac{(F^3 - 1)}{DF^2}\frac{1}{r} + \frac{(\eta + 2)(\eta - 1)}{\eta DF^2}\frac{1}{r} + \frac{D}{F}r\right], \tag{2.21}$$

and where the response coefficients are

$$\beta_i = 2\hat{G}\,\alpha_i, \qquad (i = \pm 1), \tag{2.22}$$

it being noted that,

$$\alpha_1 = \frac{3q}{2(qI_{\mathbf{B}} + II_{\mathbf{B}})} = -q\alpha_{-1}, \qquad q = \frac{dII_{\mathbf{B}}/dp}{dI_{\mathbf{B}}/dp}, \tag{2.23}$$

$$\Phi = \beta_1 - \frac{1}{\eta F}\beta_{-1} = 3\frac{(q\eta F + 1)}{(qI_{\mathbf{B}} + II_{\mathbf{B}})}\frac{1}{\eta F}\hat{G}, \tag{2.24}$$

$$\nu = \frac{3b_1'}{b_3' - b_2'} = \frac{3\rho}{(4 + \chi^2)^{1/2}}, \tag{2.25}$$

$$\bar{\nu} = \frac{3m_1'}{m_3' - m_2'} = \frac{3(M'\langle rr\rangle/M'\langle\theta z\rangle}{(4 + \chi^2)^{1/2}} = \bar{\mu}, \tag{2.26}$$

and that the b'_a, m'_a (a = 1, 2 3) are the proper numbers of $\mathbf{B}'$ and $\mathbf{M}'$. In eqns (2.22) and (2.23), both $\hat{G}$ and the α_i ($i = \pm 1$) are functions of the first and second invariants $\mathrm{I_B}$ and $\mathrm{II_B}$ or $\mathbf{B}$ only, since for an incompressible material $\mathrm{III_B} = 1$.

From eqn (2.21)

$$\zeta = -\frac{1}{3}\frac{\mu}{\bar{\mu}}\left\{\frac{(F^3-1)}{DF^2}\frac{1}{r} + \frac{(\eta+2)(\eta-1)}{\eta DF^2}\frac{1}{r} + \frac{D}{F}r - 2Q\left[\frac{D}{F}r - \frac{(\eta F^3-1)(\eta^2-1)}{\eta DF^2}\frac{1}{r}\right]\right\}, \tag{2.27}$$

where use has been made of eqn (2.26), and where

$$Q = \frac{\beta_{-1}}{\beta_{-1} - \eta F\beta_1} = \frac{1}{1+q\eta F}, \qquad q = -\beta_1/\beta_{-1}, \tag{2.28}$$

the response coefficients β_i ($i = \pm 1$) being defined by eqns (2.22) and (2.23).

2.2 *Resultant longitudinal force on plane end of a tube or solid rod*

For zero applied traction on the curved surface at $r = r_1$, such that $[\sigma_{rr}]_{r=r_1} = 0$, the condition of equilibrium in the radial direction can be expressed in the form

$$\sigma_{rr} = \int_{r_2}^{r} [(\sigma_{\theta\theta} - \sigma_{rr})/r]\,\mathrm{d}r + [\sigma_{rr}]_{r=r_2}. \tag{2.29}$$

Equation (2.29) can be used to express the resultant longitudinal normal force N_z on a plane end of a cylindrical tube in the form

$$N_z = 2\pi\int_{r_2}^{r_1} r\sigma_{zz}\,\mathrm{d}r = \frac{1}{2}\pi\int_{r_2}^{r_1}\tau_{\theta z}\left[\left(3-\frac{r_2^2}{r^2}\right)\xi - 3\left(1+\frac{r_2^2}{r^2}\right)\zeta\right]r\,\mathrm{d}r. \tag{2.30}$$

Substituting for ξ and ζ in eqn (2.30) from eqns (2.20) and (2.27) gives:

$$\begin{aligned}N_z = {}&\frac{1}{2}\pi\int_{r_2}^{r_1}\left(\frac{\tau_{\theta z}}{r}\right)\left[\frac{(F^3-1)}{DF^2} - \frac{(\eta-1)}{DF^2} - \frac{D}{F}r^2\right]\left(3-\frac{r_2^2}{r^2}\right)r\,\mathrm{d}r\\ &+\frac{1}{2}\pi\int_{r_2}^{r_1}\left(\frac{\tau_{\theta z}}{r}\right)\left(\frac{\mu}{\bar{\mu}}\right)\left[\frac{(F^3-1)}{DF^2} + \frac{(\eta+2)(\eta-1)}{\eta DF^2} + \frac{D}{F}r^2\right]\left(1+\frac{r_2^2}{r^2}\right)r\,\mathrm{d}r\\ &+\frac{1}{2}\pi\int_{r_2}^{r_1}\left(\frac{\tau_{\theta z}}{r}\right)\left(\frac{\mu}{\bar{\mu}}\right)2Q\left[\frac{(\eta F^3-1)(\eta^2-1)}{\eta DF^2} - \frac{D}{F}r^2\right]\left(1+\frac{r_2^2}{r^2}\right)r\,\mathrm{d}r\end{aligned} \tag{2.31}$$

where Q is defined by eqn (2.28).

For a solid rod, $r_2 = R_2 = 0$, $\eta = 1$, and eqn (2.31) reduces to,

$$F - 1 = \frac{2F^2D}{\pi(1+F+F^2)}\frac{N_z}{\int_0^{r_1}\left(\frac{\tau_{\theta z}}{r}\right)\left(3+\frac{\mu}{\bar{\mu}}\right)r\,\mathrm{d}r} + \frac{FD^2}{(1+F+F^2)}\frac{\int_0^{r_1}\left(\frac{\tau_{\theta z}}{r}\right)\left[3-\frac{\mu}{\bar{\mu}}(1-2Q)\right]r^3\,\mathrm{d}r}{\int_0^{r_1}\left(\frac{\tau_{\theta z}}{r}\right)\left(3+\frac{\mu}{\bar{\mu}}\right)r\,\mathrm{d}r}. \tag{2.32}$$

3. INITIAL YIELD

3.1 *Small strains*

Let the thin-walled tube be given an initial extension by means of a directly applied load W, which can be interpreted as a maximum tensile stress σ'' in simple uniaxial, that is pure tension. Partial removal of the load leaves a fraction, mW, $(0 \leq m \leq 1)$ which can be interpreted as a tensile stress $\sigma_{zz} = m\sigma''$. A stress which is effectively equivalent to the maximum tensile stress σ'' can be generated by holding this reduced load constant and applying a gradually increasing torque, thus giving rise to a shear stress τ_m at the onset of further irrecoverable extension.

For the reloading in simple torsion, the elastic strains are such that $D \to 0$, $F \to 1$, $\eta \to 1$, and eqns (2.22) and (2.23) give $\beta_i = (\hat{G}/2)i$, $(i = \pm 1)$, which in turn reduce eqn (2.28) to $Q = \frac{1}{2}$. These limiting conditions reduce eqns (2.20) and (2.27) to

$$\lim_{D\to 0} \xi = 3\frac{e_{zz}}{\gamma_{\theta z}} - \frac{(\eta - 1)}{\gamma_{\theta z}} - \gamma_{\theta z}, \tag{3.1}$$

$$\lim_{D\to 0} \zeta = -\frac{\mu}{\bar{\mu}}\left(\frac{e_{zz}}{\gamma_{\theta z}} + \frac{\eta - 1}{\gamma_{\theta z}}\right), \tag{3.2}$$

which can be combined to give

$$\frac{e_{zz}}{\gamma_{\theta z}} - \frac{1}{4}\gamma_{\theta z} = \frac{1}{12}\xi\left[3 - \bar{\mu}\left(1 + \frac{4}{\xi^2}\right)^{1/2}\right], \tag{3.3}$$

where

$$e_{zz} = F - 1, \qquad \gamma_{\theta z} = rD/F, \tag{3.4}$$

and where use has been made of eqn (2.7), it being noted that for the stress ratios ξ and ζ,

$$\sigma_{zz} = m\sigma'', \qquad (0 \leq m \leq 1). \tag{3.5}$$

Equation (3.3) is independent of the change in volume $[(\eta - 1)V]_{r=r_2}$ of the inner bore of the tube.

The conditions at initial yield are obtained from eqn (2.10) by setting $\kappa = k^2$, $[h]_{\kappa=k^2} = f$ to give

$$f = J_2' H(\omega) - k^2 = 0, \tag{3.6}$$

where f is the yield function and where k is the maximum shear stress at yielding in a state of pure shear. Three types of yield function are of interest.

(a) *Continuously differentiable yield condition*:

$$\bar{\mu} = \mu\left[1 - \frac{9c(3 + \mu^2)(1 - \mu^2)(9 - \mu^2)}{(3 + \mu^2)^3 - 8c\mu^4(9 - \mu^2)}\right], \qquad H = (1 - c\omega)\ (\equiv H_{(1)}), \tag{3.7}$$

where c is a constant characteristic of material properties. Substituting the H of eqn $(3.7)_2$ into eqn (3.6) and rearranging gives

$$\xi = \frac{2\sqrt{3}\, m b_{(1)}(1 - c\omega)^{1/2}}{\{4 - (3 + b_{(1)}^2)b_{(1)}^2 m^2 - c[4 - (3 + b_{(2)}^2)m^2 b_{(1)}^2\omega]\}^{1/2}}, \tag{3.8}$$

which can then be entered into eqn (3.3) to give,

$$\frac{e_{zz}}{\gamma_{\theta z}} - \frac{1}{4}\gamma_{\theta z} = \frac{1}{2\sqrt{3}} \frac{mb_{(1)}(1 - c\omega)^{1/2}[3 + b_{(2)}(1 - c\bar{\kappa}(\mu))]}{\{4 - (3 + b_{(2)}^2)m^2b_{(1)}^2 - c[4 - (3 + b_{(2)}^2)m^2b_{(1)}^2\omega]\}^{1/2}}, \quad (3.9)$$

where

$$b_{(1)} = 1 - \frac{\sigma_{\theta\theta}}{m\sigma''}, \qquad b_{(2)} = 1 - 2\frac{(\sigma_{rr} - \sigma_{\theta\theta})}{m\sigma'' - \sigma_{\theta\theta})}, \quad (3.10)$$

and where, from eqn $(3.7)_1$,

$$\bar{\kappa}(\mu) = \frac{9(3 + \mu^2)(1 - \mu^2)(9 - \mu^2)}{(3 + \mu^2)^3 - 8c\mu^4(9 - \mu^2)}. \quad (3.11)$$

The von Mises yield criterion is obtained by setting $c = 0$, thus giving $H = 1$. For the infinitely small strains up to and including the initial yield of many metals, it is customary to assume that $\sigma_{rr} \simeq 0$, $\sigma_{\theta\theta} \simeq 0$. These limiting conditions reduce eqns (3.8) and (3.9) for the von Mises yield criterion to the form

$$\xi = \frac{\sqrt{3}\,m}{(1 - m^2)^{1/2}}, \qquad \frac{e_{zz}}{\gamma_{\theta z}} - \frac{1}{4}\gamma_{\theta z} = \frac{1}{\sqrt{3}}\frac{m}{(1 - m^2)^{1/2}}. \quad (3.12)$$

For $m = 0$, that is simple torsion, both von Mises yield condition and its modified form give

$$e_{zz} = \frac{1}{4}\gamma_{\theta z}^2, \quad (3.13)$$

from which it follows that $e_{zz}/\gamma_{\theta z} = 0$ for $\gamma_{\theta z} = 0$.

(b) *Piece-wise continuous yield condition*:

$$\bar{\mu} = \text{const.}, \; H = \frac{(3 + \bar{\mu}_n\mu)^2}{(3 + \mu^2)}\left[\frac{(3 + \mu^2)}{(3 + \bar{\mu}_n\mu)^2}\right]_{\omega=0} (\equiv H_{(2)}),$$

$$n = \begin{cases} 0, & (0 \leq \mu^2 \leq 1) \\ 2, & (1 \leq \mu^2 \leq 9) \\ -2, & (9 \leq \mu^2 \leq \infty) \end{cases}. \quad (3.14)$$

Substituting the H of eqn $(3.14)_2$ into eqn (3.6) and rearranging gives

$$\xi = \frac{6mb_{(1)}}{\{[3 + |\bar{\mu}_0|(1 - mb_{(1)}b_{(2)})]^2 - 9m^2b_{(1)}^2\}^{1/2}}, \quad (3.15)$$

which can then be entered into eqn (3.3) to give,

$$\frac{e_{zz}}{\gamma_{\theta z}} - \frac{1}{4}\gamma_{\theta z} = \frac{1}{6}\frac{\{9mb_{(1)} + |\bar{\mu}_0|[3 + |\bar{\mu}_0|(1 - mb_{(1)}b_{(2)})]\}}{\{[3 + |\bar{\mu}_0|(1 - mb_{(1)}b_{(2)})]^2 - 9m^2b_{(1)}^2\}^{1/2}}. \quad (3.16)$$

Equations (3.1) and (3.2) can be combined to give

$$2\frac{(\sigma_{rr} - \sigma_{\theta\theta})}{\tau_{\theta z}} = 3\left(1 - \frac{\mu}{\bar{\mu}_0}\right)\frac{e_{zz}}{\gamma_{\theta z}} - \left(1 + 3\frac{\mu}{\bar{\mu}_0}\right)\frac{(\eta - 1)}{\gamma_{\theta z}} - \gamma_{\theta z} = 3\zeta + \xi. \quad (3.17)$$

For sufficiently small strains, and for $m \to 1$, $\sigma_{rr} \to 0$, $\sigma_{\theta\theta} \to 0$, the stress ratios $(\sigma_{rr} - \sigma_{\theta\theta})/\tau_{\theta z}$ and $\sigma_{\theta\theta}/m\sigma''$ can both be neglected, giving

$$b_{(1)} = 1, \quad b_{(2)} = 1, \quad \xi = -3\zeta, \quad (\sigma_{rr} = 0, \sigma_{\theta\theta} = 0); \tag{3.18}$$

these conditions reduce eqns (3.15) and (3.16) to

$$\xi = \frac{6m}{\{[3 + |\bar{\mu}_0|(1 - m)]^2 - 9m^2\}^{1/2}}, \quad (\sigma_{rr} = 0, \sigma_{\theta\theta} = 0), \tag{3.19}$$

$$\frac{e_{zz}}{\gamma_{\theta z}} - \frac{1}{4}\gamma_{\theta z} = \frac{1}{6}\frac{\{9m + |\bar{\mu}_0|[3 + |\bar{\mu}_0|(1 - m)]\}}{\{[3 + |\bar{\mu}_0|(1 - m)]^2 - 9m^2\}^{1/2}}. \tag{3.20}$$

As m is increased from zero, the effect of the applied torque is rapidly attenuated, thus giving way to the effect of the increasing applied tensile stress. There will, therefore, be a particular value of $m = m^* \ll 1$, above which eqn (3.20) will be expected to apply.

The limiting condition $m = 0$ corresponds to simple torsion for which $F \geq 1$. Setting $m = 0$ reduces eqn (3.16) to

$$\frac{e_{zz}}{\gamma_{\theta z}} - \frac{1}{4}\gamma_{\theta z} = \frac{1}{6}\frac{\left\{|\bar{\mu}_0|\left[3 + |\bar{\mu}_0|\left(1 - \frac{\sigma_{\theta\theta} - 2\sigma_{rr}}{\sigma''}\right)\right] - 9\frac{\sigma_{\theta\theta}}{\sigma''}\right\}}{\left\{\left[3 + |\bar{\mu}_0|\left(1 - \frac{\sigma_{\theta\theta} - 2\sigma_{rr}}{\sigma''}\right)\right]^2 - 9\frac{\sigma_{\theta\theta}^2}{\sigma''^2}\right\}^{1/2}} \tag{3.21}$$

and for $\sigma_{\theta\theta} = 0$ at $\gamma_{\theta z} = 0$, eqn (3.21) gives

$$\left[\frac{e_{zz}}{\gamma_{\theta z}}\right]_{\gamma_{\theta z}=0} = \frac{1}{6}|\bar{\mu}_0|, \quad (m = 0, \sigma_{\theta\theta} = 0). \tag{3.22}$$

Thus, for a material to satisfy a 12-sided, piece-wise continuous yield condition, for which $|\bar{\mu}_0| =$ const. for all μ, the material must respond to simple torsion in such a way as to give $e_{zz}/\gamma_{\theta z} =$ const., at $\gamma_{\theta z} = 0$. This is a limiting condition for which there is no experimental evidence.

(c) *Composite yield function*:

$$H = \phi^* H_{(2)} + (H_{(1)} - \phi^* H_{(2)})U, \tag{3.23}$$

$$\phi^* = 3\frac{(1 - c\bar{\omega})}{(3 + \bar{\mu}_0^2)}, \quad U = \begin{cases} 1, & (0 \leq \mu^2 \leq \bar{\mu}_0^2) \\ 0 & (\bar{\mu}_0^2 \leq \mu^2 \leq 1) \end{cases}, \quad (\omega = \bar{\omega}, \mu = \bar{\mu}_0). \tag{3.24}$$

A material of particular interest is one which satisfies the yield condition obtained using eqns (3.6) and (3.23), the transition from one type of yield condition to the other occurring in such a way that the ϕ^* of eqn $(3.24)_1$ is to be identified with the value of $\mu = \bar{\mu}_0$ at which $m = m^* \ll 1$, and above which eqn (3.20) will be expected to apply. For this type of composite yield condition, eqn (3.9) applies for $(0 \leq m \leq m^*)$. In this way the composite yield function satisfies the essential condition for $m = 0$, $e_{zz}/\gamma_{\theta z} = 0$ at $\gamma_{\theta z} = 0$.

3.2 *Material response*

The programme of combined stressing described by Taylor and Quinney[3] is just that considered above. Central to the combined stress measurements of Taylor and Quinney is the quantity

$$\frac{2}{\xi} = \frac{2\tau_{\theta z}}{\sigma_{zz} - \sigma_{\theta\theta}}, \quad (\equiv \tan 2\theta \text{ in their notation}). \tag{3.25}$$

It is therefore assumed that Taylor and Quinney will have determined this quantity as accurately as their measurements permit, it being noted that ξ is defined by eqn (2.5). The values of $2/\xi$ have been recalculated and, with the exception of $m = 0.90$ for the copper tubes, found to be in good accord with the values given by Taylor and Quinney in their Table 1. Taking their value of $2/\xi$, recalculation gives $m = 0.886$, which, to the degree of accuracy of the original analysis, is taken to imply the value 0.89 which is used in the present discussion.

In the context of eqn (3.5), the maximum tensile stress σ'' is the initial yield stress Y in simple uniaxial tension. Values of Y ($\equiv \sigma''$) and σ_{zz} are given in the present Table 1 for the copper tubes tested by Taylor and Quinney[3].

The effect of applying the load W is to extend the tube from its initial length L to a length l_0; hence from eqn (2.16), $F_0 = l_0/L$ ($\equiv 1 + e_0$ in the notation of Taylor and Quinney).

Partial removal of the load leaves a fraction mW, which for a given tube can be interpreted as the tensile stress $\sigma_{zz} = mY$, defined by eqn (3.5) with σ'' set equal to Y.

For any given value of m, the reloading in torsion gives to a good approximation a constant value for the quantity

$$\psi^* = \frac{LF_0D}{\delta l}\left(\equiv \frac{\chi}{\delta l}, \text{ in their notation}\right), \tag{3.26}$$

where δl is the additional extension of the tube resulting from application of an increasing torque. This gives rise to an axial strain

$$e_{zz} = \frac{(1 + e_0)^{1/2}}{\psi^* R_{m0}}\gamma_{\theta z}, \tag{3.27}$$

where R_{m0} is the mean radius of the tube before the first stretching, and where $\gamma_{\theta z}$ is the shear strain. Using standard data-retrieval techniques, the actual values of the total angle of twist LF_0D and the extension of the tube δl have been obtained from Fig. 7(a) of Taylor and Quinney[3]. The (LF_0D, δl) results have been analysed, assuming the linear relation of eqn (3.26) but using only the measured values of LF_0D and δl. The values of ψ^* obtained in this way have been used with eqn (3.27) to give the values of the ratio $e_{zz}/\gamma_{\theta z}$ shown in the present Table 1. The variation of these values of $e_{zz}/\gamma_{\theta z}$ with m are shown as open symbols in Fig. 1. Using the same approach, values of the ratio $e_{zz}/\gamma_{\theta z}$ for the aluminium tubes have been obtained from Fig. 7(b) of Taylor and Quinney[3], and are given in Table 2. The variation of these values of $e_{zz}/\gamma_{\theta z}$ with m are also shown as open symbols in Fig. 1, the experimental ($e_{zz}/\gamma_{\theta z}$, m) curve being displaced by 0.2 units parallel to the ordinate.

For the aluminium and copper tubes, $(\gamma_{\theta z}/4) \lll (e_{zz}/\gamma_{\theta z})$ for $m \geqslant 0.25$ and hence $\gamma_{\theta z}/4$ can be neglected on the left-hand side of eqns $(3.12)_2$ and (3.20). Using this approximation, and the value $\bar{\mu}_0 = -0.305$ for the range ($m^* \leqslant m \leqslant 1$), the full line ($e_{zz}/\gamma_{\theta z}$, m) curve shown in Fig. 1 has been calculated for the copper tubes from eqn (3.20) and is seen to be indistinguishable from the experimental ($e_{zz}/\gamma_{\theta z}$, m) curve. The values of

Table 1. Combined stress measurements using the copper tubes of Taylor and Quinney (1931)

m	0.0252	0.280	0.515	0.65	0.70	0.80	0.89
Y (MPa)	217.1	222.9	216.5	216.5	180.3	218.0	222.9
σ_{zz} (MPa)	5.47	62.4	111.5	140.7	126.2	174.4	198.4
†$e_{zz}/\gamma_{\theta z}$	0.0155	0.166	0.340	0.468	0.530	0.717	1.033
‡$e_{zz}/\gamma_{\theta z}$	0.0146	0.168	0.340	0.468	0.532	0.714	1.034
$-\bar{\mu}$	0.0232	0.2416	0.3049	0.3018	0.2900	0.3280	0.2988
$-\mu$	0.0219	0.2444	0.4908	0.6279	0.6803	0.7829	0.8804

† Taylor & Quinney (1931)
‡ From eqns (3.12) and (3.20)

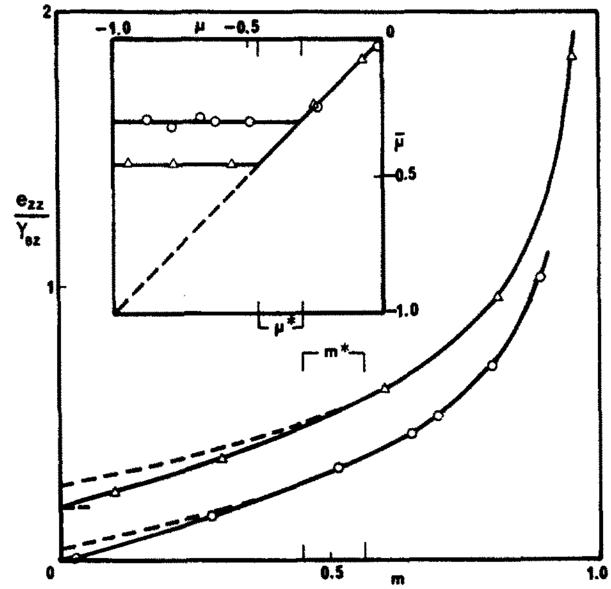

Fig. 1. Variation of $e_{zz}/\gamma_{\theta z}$ with m for the combined stress measurements of Taylor and Quinney[3] using their results for copper (O) and aluminium (Δ) tubes.

$e_{zz}/\gamma_{\theta z}$ calculated in this way for $m \geq m^* = 0.45$ are given in Table 1 for direct comparison with the values of the ratio $e_{zz}/\gamma_{\theta z}$ determined from the measurements of Taylor and Quinney[3]. The calculated extension of the $(e_{zz}/\gamma_{\theta z}, m)$ curve for $(0 \leq m \leq m^*)$ is shown as the broken line curve in Fig. 1. The full-line $(e_{zz}/\gamma_{\theta z}, m)$ curve shown in Fig. 1 for the aluminium tubes has been calculated from eqn (3.20) using $\bar{\mu}_0 = -0.454$ and is seen to be indistinguishable from the experimental $(e_{zz}/\gamma_{\theta z}, m)$ curve for $m \geq m^* = 0.563$. The calculated values of $e_{zz}/\gamma_{\theta z}$ for $m \geq m^*$ are given in Table 2 for direct comparison with the values of the ratio $e_{zz}/\gamma_{\theta z}$ determined from experiment. The calculated extension of the $(e_{zz}/\gamma_{\theta z}, m)$ curve is shown as the broken–line curve in Fig. 1. It is concluded that the combined-stress measurements of Taylor and Quinney[3] can be accurately correlated for m in the range $(m^* \leq m \leq 1)$ by the 12-sided, linear, piece-wise continuous yield condition defined by eqn (3.6) with H given by eqn $(3.14)_2$.

Neglecting the $\gamma_{\theta z}/4$ term, the full–line $(e_{zz}/\gamma_{\theta z}, m)$ curves shown in Fig. 1 have been calculated for the range $(0 \leq m \leq m^*)$ from eqn $(3.12)_2$ for the limiting conditions $\sigma_{rr} = 0$, $\sigma_{\theta\theta} = 0$, and is seen to be indistinguishable from the experimental $(e_{zz}/\gamma_{\theta z}, m)$ curves for both the copper and aluminium tubes. The values of $e_{zz}/\gamma_{\theta z}$ calculated in this way for $m \leq m^*$ are given in Tables 1 and 2 for direct comparison with the values of the ratio $e_{zz}/\gamma_{\theta z}$ determined from the measurements of Taylor and Quinney[3]. It is concluded that the combined-stress measurements of Taylor and Quinney can be correlated for m in the range $(0 \leq m \leq m^*)$ by the von Mises yield criterion defined by eqn (3.6) with $H = 1$ for all μ.

Table 2. Combined stress measurements using the aluminum tubes of Taylor and Quinney (1931)

m	0.10	0.30	0.60	0.81	0.95
Y (MPa)	93.8	93.3	94.6	94.6	93.8
σ_{zz} (MPa)	9.38	28.0	56.8	76.7	89.1
†$e_{zz}/\gamma_{\theta z}$	0.050	0.166	0.435	0.761	1.641
‡$e_{zz}/\gamma_{\theta z}$	0.058	0.182	0.435	0.761	1.642
$-\bar{\mu}$	0.0748	0.2416	0.4540	0.4540	0.4498
$-\mu$	0.0867	0.2634	0.5657	0.7875	0.9429

† Taylor & Quinney (1931)
‡ From eqns (3.12) and (3.20)

It is of particular interest to note that this reanalysis of the combined-stress measurements of Taylor and Quinney[3] does not in any way depend on the change in volume, $[(\eta - 1)V]_{r=r_2}$, of the inner bore of the tubular specimen; furthermore, it does not require a measurement of the shear stress τ_m corresponding to the onset of plastic distortion.

Values of $\bar{\mu}$ for m in the range ($m^* \leq m \leq 1$) have been calculated from eqn (3.20) using the experimental values of the ratio $e_{zz}/\gamma_{\theta z}$ determined by Taylor and Quinney and given in the present Tables 1 and 2. Using these values of $\bar{\mu}$ ($\equiv \bar{\mu}_0$), corresponding values of ξ have been calculated from eqn (3.19), these values of ξ being substituted into eqn (2.7) to give values of μ for the limited condition $\xi = -3\zeta$ of eqn (3.18). These values of $\bar{\mu}$($\equiv \bar{\mu}_0$) and μ are given in Tables 1 and 2 and are shown as the (μ, $\bar{\mu}$) curves in the inset diagram to Fig. 1.

For m in the range ($0 \leq m \leq m^*$), the values of ξ have been calculated from the relation of eqn $(3.12)_1$ and then substituted into eqn (2.7) to give values of μ for the limiting condition $\xi = -3\zeta$ of eqn (3.18). Corresponding values of $\bar{\mu} = \mu$ have been evaluated by way of eqns (2.7) and (3.18) and the relation of eqn $(3.12)_1$, using the experimental values of the ratio $e_{zz}/\gamma_{\theta z}$ determined by Taylor and Quinney and given in the present Tables 1 and 2. These values of $\mu = \bar{\mu}$ for $m \leq m^*$ are given in Tables 1 and 2 and are shown as the (μ, $\bar{\mu}$) curves in the inset diagram to Fig. 1.

The form of the (μ, $\bar{\mu}$) relation shown in the inset diagram of Fig. 1 is taken to imply that the material of the copper and aluminium tubes satisfies a composite yield function defined by eqns (3.6) and (3.23). The yield surface in the principal stress space is shown in Fig. 2, that is the yield curve in the deviatoric plane, (for a discussion of the yield surface, see for example, Billington[8]).

4. POYNTING EFFECT

The Poynting effect is characterised by an axial elongation

$$e_{zz} = F - 1 \;(\equiv e_{(p)}), \tag{4.1}$$

which is observed for the particular condition, $N_z = 0$.

For sufficiently small strains, for which $D \to 0$, $F \to 1$, $\eta \to 1$, eqns (2.22) and (2.23) give $\beta_i = (\hat{G}/2)\, i$, $(i = \pm 1)$, which in turn reduce eqn (2.28) to $Q = \frac{1}{2}$.

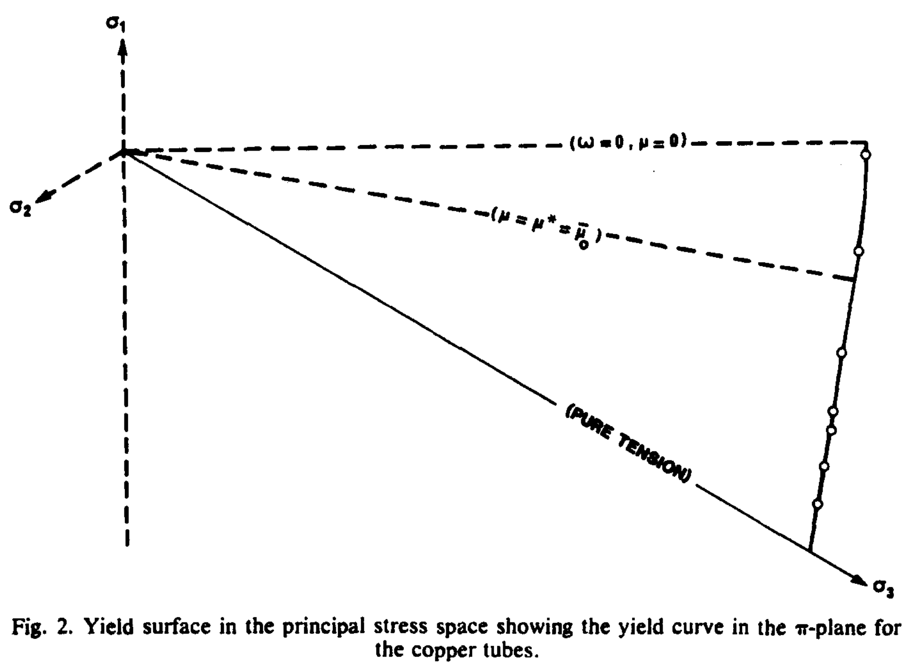

Fig. 2. Yield surface in the principal stress space showing the yield curve in the π-plane for the copper tubes.

The limiting conditions for small strains reduce eqn (2.32) to,

$$e_{(p)} = a\gamma_{\theta z} + b\gamma_{\theta z}^2 \tag{4.2}$$

where the shear strain

$$\gamma_{\theta z} = R_1 D, \qquad (F \to 1), \tag{4.3}$$

and where

$$a = \frac{2N_z}{3\pi R_1 \int_0^{r_1} \left(\frac{\tau_{\theta z}}{r}\right)\left(3 + \frac{\mu}{\bar{\mu}}\right) r\,dr}, \qquad b = \frac{1}{R_1^2} \frac{\int_0^{r_1} \left(\frac{\tau_{\theta z}}{r}\right) r^3\,dr}{\int_0^{r_1} \left(\frac{\tau_{\theta z}}{r}\right)\left(3 + \frac{\mu}{\bar{\mu}}\right) r\,dr}. \tag{4.4}$$

For the special case that the ratios $\tau_{\theta z}/r$ and $\mu/\bar{\mu}$ are both independent of r,

$$b = \frac{r_1^2}{2R_1^2\,(3 + \mu/\bar{\mu})}. \tag{4.5}$$

Thus, in the absence of any dependence upon r, and noting that for a stress–intensity function based on the von Mises[1] yield criterion, $\mu = \bar{\mu}$ for all μ, the ratio b should have the limiting value

$$\lim_{\gamma_{\theta z} \to 0} b = \frac{1}{8}, \tag{4.6}$$

for a solid rod specimen of a material twisted in simple torsion and which satisfies a stress intensity function based on the von Mises yield criterion.

The results of the experimental studies of Poynting[4, 5], Foux[9], and of Lenoe *et al.*[10] are in good accord with the prediction that in simple torsion, the associated axial extension $e_{(p)}$ is proportional to $\gamma_{\theta z}^2$ in the elastic range of deformation. However, there remains the question of the extent to which the experimental work is in quantitative agreement with the predictions of nonlinear constitutive theory, in particular the value of the ratio $e_{(p)}/\gamma_{\theta z}^2$.

Foux[9] analysed his extensive test series on hard steel wires in simple torsion assuming that the axial elongation could be expressed as a polynomial expansion in $\gamma_{\theta z}$. For present purposes it will be assumed that

$$e_{(p)} = a_0 + a\gamma_{\theta z} + b\gamma_{\theta z}^2, \tag{4.7}$$

where a_0, a and b are regarded as constants, eqn (4.7) differing from eqn (4.2) by the presence of the arbitrary constant a_0 which has been introduced to take account of any slight lack of symmetry in the ($e_{(p)}$, $\gamma_{\theta z}$) curve arising out of the intrinsic limits of experimental accuracy. The reason for retaining the term a, which arises from N_z and is defined by the left–hand relation of eqn (4.4), is that although the weight of the apparatus may have negligible effect in pure tension, the question of whether it changes the torsional modulus in the simple torsion mode of deformation can not be resolved. The presence of the term in a_0 necessitates a slightly different interpretation of the measurements given by Foux[9] in his Table 2. For present purposes, the positive and negative cycles have been independently analysed to give two values of b. The variation of the average of these two values of b with $(R_1/L)^2$ is shown in Fig. 3 for wire numbers 4, 6, 7 and 9, in the as-drawn state and after heating. Wire numbers 2, 5 and 8 are not shown because they are from different sources. From an examination of the dependence of the values of b on E and the ultimate tensile strength, it is concluded that wire numbers 1 and 3 are in a significantly different material state and are therefore also omitted from Fig. 3. Wire number 10 is omitted because the effect of reversing the direction of shear of the as-received specimen is to reverse the direction of the change in length of the wire. This implies that for this radius of wire, the material properties are markedly different, and hence there is the possibility that heat treatment will not necessarily produce the required change in material properties. It is evident from Fig. 3 that the values of b are dependent upon the cross-section of the wires. There are two possible explanations for such a dependence upon r. This type of measurement is subject to an error arising from an apparent strengthening by the material nearer the longitudinal axis of the specimen which is stressed to a lower level. A second contributing factor is the linear dimension of the grains which for some materials may approach that of the diameter of the wire. Because of these limitations, and having regard to the extent of the extrapolation, these results do not give a conclusive value for b. However, these results do give values of b which approximate quite closely to the value 0.125, (c.f eqn (4.6)), characteristic of a material which satisfies a stress intensity function based on von Mises[1] yield criterion, [see Section 3.1(a)].

The nonlinear mechanical response of polyurethane rubbers with a high volume percentage of inorganic filler has been studied by Lenoe *et al.*[10]. Values of $b = 0.14$ at a strain rate of 0.00013 s^{-1} and of $b = 0.095$ at a strain rate of 0.0013 s^{-1} were obtained in this study using solid–rod cylindrical specimens tested in simple torsion at the given constant strain rates and for a value of $(R_1/L)^2 = 0.04$. Since these values of b also closely approximate that given by eqn (4.6) for a material which satisfies a stress–intensity function based on von Mises[1] yield criterion, and having regard to the very large diameter of these specimens compared with that of the metal wires studied by Foux[9], it is concluded that these rubber specimens exhibit only a very small dependence upon r.

5. POYNTING-SWIFT EFFECT

5.1 *Small strains*

The Poynting-Swift effect is characterised by an axial elongation,

$$e_{zz} = F - 1 \; (\equiv e_{(ps)}), \tag{5.1}$$

which is observed for the particular condition, $N_z = 0$.

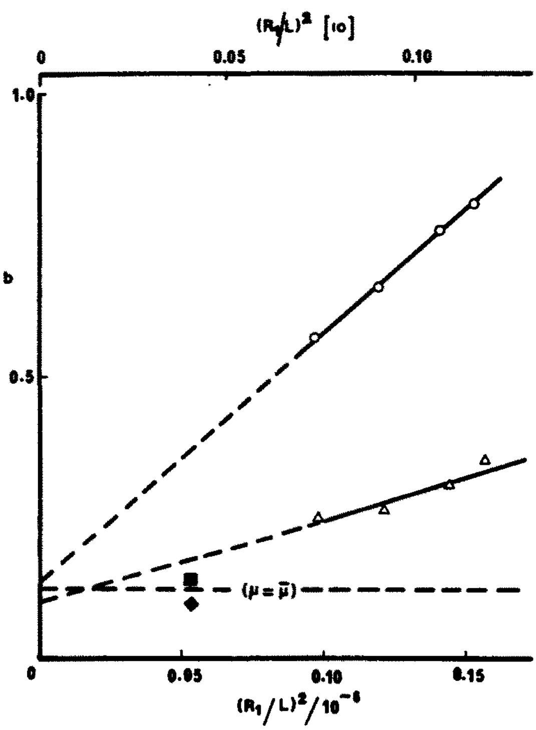

Fig. 3. Variation of b with $(R_1/L)^2$. ○, △[9]. $\dot{\gamma}$ (s^{-1}): ■, 0.00013; ◆, 0.0013,[10].

For sufficiently small strains, for which $D \to 0$, $F \to 1$, $\eta \to 1$, eqns (2.22) and (2.23) give $\beta_i = (\hat{G}/2)i$, $(i = \pm 1)$, which in turn reduces eqn (2.28) to $Q = \frac{1}{2}$.

With $N_z = 0$, the limiting conditions for small strains, together with the condition that the cylinders $R = R_1$ and $R = R_2$ deform into cylinders free of traction, reduce eqn (2.31) to a form which can be integrated, (subject to the further condition that for sufficiently thin-walled tubes the ratios $\tau_{\theta z}/r$ and $\mu/\bar{\mu}$ can be regarded as being independent of r), to give

$$e_{(ps)} = \frac{1}{\left(3 + \dfrac{\mu}{\bar{\mu}}\right)} \left[\frac{2(r_1^2 + r_2^2)}{(r_1 + r_2)^2}\right] \gamma_{\theta z}^2, \tag{5.2}$$

where the shear strain,

$$\gamma_{\theta z} = (R_1 + R_2)\, D/2. \tag{5.3}$$

For a stress–intensity function based on the von Mises yield criterion, $\mu = \bar{\mu}$ for all μ, and eqn (5.2) gives for a sufficiently thin-walled tube

$$e_{(ps)} = \frac{1}{4}\gamma_{\theta z}^2, \tag{5.4}$$

for an initial, limited range of $\gamma_{\theta z}$.

The way in which the total, static axial strain $e_{(ps)}$ varies with the shear strain $\gamma_{\theta z}$ is shown in Fig. 4 for a thin-walled tubular specimen of aluminium provided with a close–fitting plug. Although the plug is close fitting, there is a small gap between the outer surface of the plug and the inner surface of the tube which is packed with an extreme pressure lubricant prior to test. The lubricant does not restrict the initial free movement of the inner surface of the tube in a radial direction. Hence the inner surface of the thin-walled tubular specimen can be regarded as being free of traction, and the material should therefore respond according to eqn (5.4) for an initial, limited range of $\gamma_{\theta z}$.

All the materials of present interest when tested in the fully annealed state, as for example the results for copper, [see eqn (5.1) and Table 1 of Billington[11]], satisfy the relation of eqn (5.4) over an initial, limited range of $\gamma_{\theta z}$. However, all the earlier test specimens of materials in the as-received state exhibited a marked change in the characteristic $(e_{(ps)}, \gamma_{\theta z})$ curve at a particular value of $\gamma_{\theta z} = \gamma_a$, (see Figs. 4 and 5 of [11]). It was observed, however, that for $\gamma_{\theta z}$ just in excess of γ_a, there was a limited range of $\gamma_{\theta z}$ over which all materials in the as-received state satisfied eqn (5.4), (see Fig. 6 of [11]). It has been observed that specimens of as-received material which have been subjected to a small permanent forward twist, giving $\gamma_{\theta z} \lll 0.01$, followed immediately by an identical reverse twist do not generally exhibit any discontinuity in the $(e_{(ps)}, \gamma_{\theta z})$ curve in the region of initial yield. Shown in the inset diagram of Fig. 4 is the much expanded initial region of the $(e_{(ps)}, \gamma_{\theta z})$ curve shown in Fig. 4. The full–line curve has been calculated from eqn (5.4) and is seen to provide a good correlation to the experimental $(e_{(ps)}, \gamma_{\theta z})$ curve over an initial limited range of $\gamma_{\theta z}$. Reviewing the earlier measurements and those of present interest it is concluded that over an initial limited range of simple torsion, the materials satisfy a stress intensity function based on the von Mises yield criterion.

5.2 *Finite strains*

Values of $\bar{\mu}$ can be evaluated from eqn (2.26) expressed in the form

$$\bar{\mu} = \frac{3(\beta_1 + \eta^2F^2\beta_{-1}) - \eta F(\beta_1 \mathrm{I_B} + \beta_{-1}\mathrm{II_B})}{(\eta F\beta_1 - \beta_{-1})F^2(4 + \chi^2)^{1/2}\,\gamma_{\theta z}}, \tag{5.5}$$

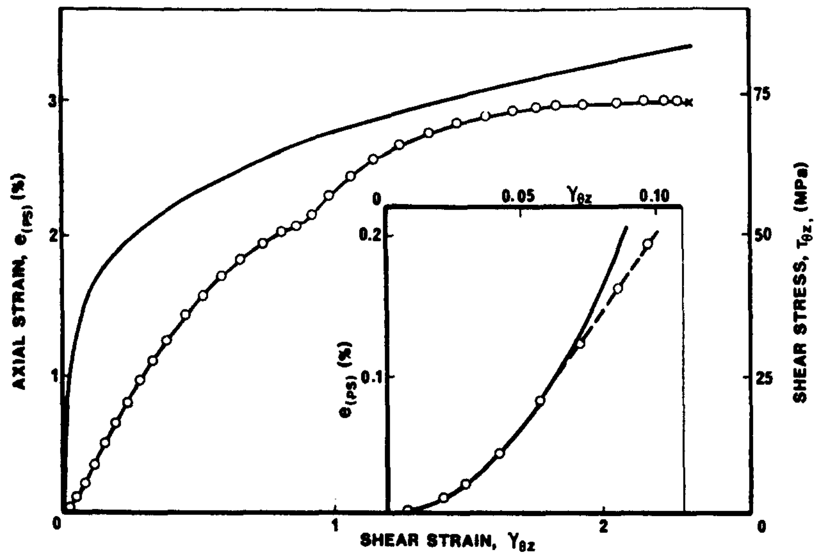

Fig. 4. Variation of the static axial strain $e_{(ps)}$ with the static shear strain $\gamma_{\theta z}$ for a thin-walled specimen of aluminium provided with a close fitting plug. Also shown is the static stress–strain ($\tau_{\theta z}$, $\gamma_{\theta z}$) curve for aluminium in simple torsion. x-denotes fracture.

where $\gamma_{\theta z} = (r_1 + r_2)D/2$, and where the β_i ($i = \pm 1$) are defined by eqns (2.22) and (2.23).

For sufficiently thin-walled tubes it will be assumed that the ratios $\tau_{\theta z}/r$ and $\mu/\bar{\mu}$ can be regarded as being independent of r. This approximation enables values of the ratio $\mu/\bar{\mu}$ to be evaluated from eqn (2.31). Values of μ can then be evaluated by use of corresponding values of $\bar{\mu}$.

The quantities $\bar{\mu}$ and $\mu/\bar{\mu}$ can only be evaluated if the dependence of η upon $\gamma_{\theta z}$ is known. The use of thin-walled tubes provided with a close-fitting plug for the purpose of evaluating η has been described by Billington[11]. Except for a small initial range of $\gamma_{\theta z}$, the inner wall of the tube is effectively in contact with the outer surface of the plug. For such specimens, $r_2 \simeq R_2$ for almost all D, and hence from eqn (2.17),

$$\eta \simeq 1 + \frac{R_2^2}{R^2}(F - 1) = 1 + \frac{R_2^2}{R^2} e_{(ps)} = 1 + \frac{K}{R^2}, \tag{5.6}$$

where $e_{(ps)}$ is the axial elongation characteristic of the Poynting-Swift effect.

It is important to note that to obtain values of μ and $\bar{\mu}$, the associated ($e_{(ps)}$, $\gamma_{\theta z}$) curve characteristic of the Poynting-Swift effect must be available, since corresponding values of F can only be obtained from the ($e_{(ps)}$, $\gamma_{\theta z}$) curve. It is in this way that the stress ratios are effectively evaluated in the context of eqns (2.20), (2.21) and (2.27), and hence all non-zero components of stress are effectively, though indirectly, accounted for.

Many of the earlier test specimens of low-strength materials, as for example aluminium and copper, exhibited a maximum in the ($e_{(ps)}$, $\gamma_{\theta z}$) curve [11]. Several of the test specimens of these particular metals formed helical folds in the tubular section, the fold developing along the direction of the helix associated with the strain $\gamma_{\theta z}$. It has been observed that there is a minimum wall thickness of tubular specimens of low-strength materials above which the ($e_{(ps)}$, $\gamma_{\theta z}$) curve does not pass through a maximum, and in particular does not form a fold prior to failure. Thus, from this observation it is tentatively concluded that the maximum in the ($e_{(ps)}$, $\gamma_{\theta z}$) curve arises out of the formation of a fold. The present studies of the Swift effect were initially based on the use of tubular specimens of the same wall thickness for all materials, the wall thickness being made as small as possible, consistent with uniformity of material properties across the width of the tube. It has been shown, however,[12] that there is a range of wall thickness over which the thin-walled tube approximation can be applied. The present

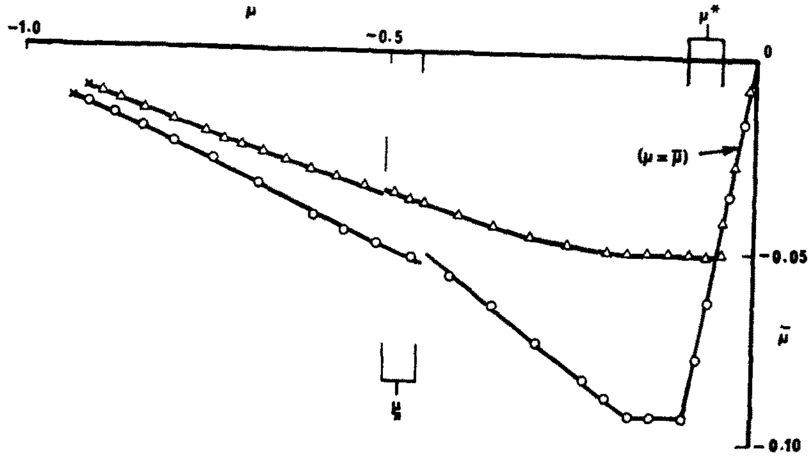

Fig. 5. Values of the parameters μ and $\bar{\mu}$ for thin-walled specimens of Al, O, and fully annealed Cu, △, determined from measurements in simple torsion.

discussion is therefore restricted to specimens having a minimum wall thickness which is determined by the condition that the specimen must not form a fold prior to failure. In this connection, the comparatively high strength materials, as for example iron, ([11], Fig. 3) do not form folds, their minimum wall thickness being determined solely by the need for uniformity of material properties across the wall.

Values of μ and $\bar{\mu}$, evaluated in the way described above, are shown in Figs. 5 and 6 as the (μ, $\bar{\mu}$) curves for specimens of aluminium, copper and iron, using the ($e_{(ps)}$, $\gamma_{\theta z}$) curve characteristic of the Poynting-Swift effect for these materials. Over an initial range of μ, these (μ, $\bar{\mu}$) curves exhibit a marked formal similarity to the curves obtained from the measurements of Taylor and Quinney[3], and which are shown in the inset diagram to Fig. 1.

It is evident from Fig. 5 that for copper in the fully annealed state there is a range ($-0.25 \leq \mu \leq -0.05$) over which $\bar{\mu}$ can be regarded as being a constant, a condition which is in accord with the assumption of a twelve-sided, piece-wise continuous yield condition. Thus over the limited range ($-0.25 \leq \mu \leq 0$), the yield surface simply expands without change in shape in accord with the concepts underlying classical isotropic hardening. For μ in excess of $\mu = -0.25$, both μ and $\bar{\mu}$ vary with the parameter κ which has been introduced into the stress intensity function of eqn (2.10) to take account of the work-hardening properties of the material. Thus, for $\mu \leq -0.25$, each value of $\bar{\mu}$ corresponds to a different state of hardness of the material. However, having regard to the combined-stress measurements of Taylor and Quin-

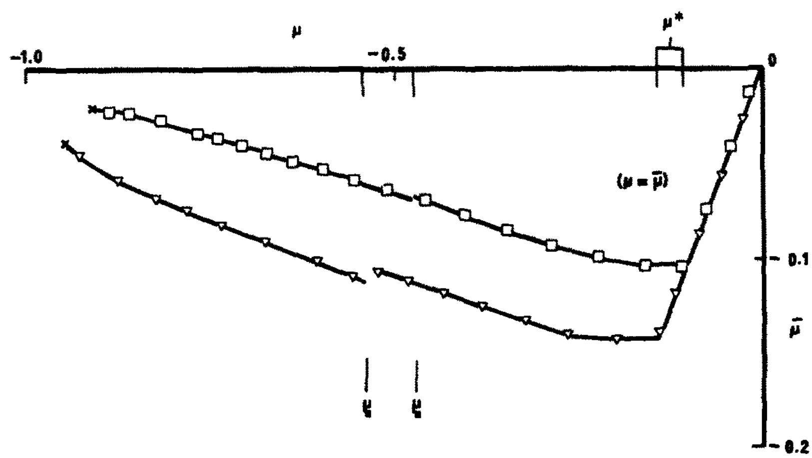

Fig. 6. Values of the parameters μ and $\bar{\mu}$ for thin-walled specimens of Cu, □, and Fe, ▽, both in the as-received state, determined from measurements in simple torsion.

ney[3], in particular the inset diagram to Fig. 1, it is concluded that for $\mu \leq -0.25$, each value of $\bar{\mu}$ corresponds to a different 12-sided, piece-wise, continuous yield condition. The same interpretation is considered to apply to the other three (μ, $\bar{\mu}$) curves shown in Figs. 5 and 6.

6. CONCLUSIONS

Combined extension and torsion of a thin-walled tube has been considered in relation to material behaviour at small strains, and in relation to both the Poynting effect and the combined Poynting-Swift effect for large shear strains. For sufficiently small shear strains it has been shown that simple torsion of both a thin-walled tube and a solid rod is characterised by an axial elongation which is proportional to the square of the twist. The ratio of the axial elongation to the square of the twist for a thin-walled tube is twice that for a solid rod.

From the present study of the Poynting-Swift effect and the reassessment of the classical combined-stress measurements of Taylor and Quinney[3], it is concluded that there are two distinct ranges of deformation. One range of deformation has the pure shear mode of deformation as one of its limits; the other range of deformation has the pure tensile, (or compressive) mode of deformation as a limit. It has been shown that the response of a material to deformation which has pure shear as one of its limits can be described by the von Mises yield criterion. For the other range of deformation the material's response can be described by the use of a 12-sided, linear, piece-wise continuous yield condition. With these observations as a basis, a composite yield function has been formulated which, over a given range of application, is identical to the von Mises yield criterion and over the remaining range of application takes the form of a 12-sided, linear, piece-wise continuous yield condition. This composite yield condition describes continuously the entire range of deformation covered by the present studies of the Poynting-Swift effect and the entire range of the combined-stress measurements of Taylor and Quinney[3].

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